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Fritzsch-Stech Model in SO(10)

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Spreading misinformation about the mass of neutrinos

Abstract

The combined Fritzsch-Stech model for quarks is consistent with our present knowledge of quark masses and mixing angles. A possible way to extend it to leptons - by using an SO(10) framework - has been suggested by Bottino, Kim, Nishiura and Sze. We investigate this model further and find some new features. It is first noted that the allowed quark masses are somewhat sensitive to the experimental uncertainties in the input parameters. This is carefully analyzed and we find that the original Stech ansatz for the hadron sector is consistent. However, if the top quark mass is greater than about 45 GeV, this ansatz must be slightly modified. It is found that in a model of the present type the see-saw mechanism for neutrino masses may be drastically modified. This occurs because in SO(10) it is likely that the matrix of heavy "right-handed" neutrinos is close to a singular matrix. This feature is analyzed both analytically and numerically. Various patterns involving two extremely light neutrinos and one intermediate mass (order of several MeV) neutrino which may arise are compared with experiment. More orthodox and perhaps more probable patterns for neutrino masses are also discussed. The lepton mixing matrices are predicted and the results support recent explanations of the solar neutrino paradox.

1. Introduction and Summary

Recently, a model[1-5] was proposed for the fundamental up and down quark mass matrices which has the property of fixing (for any number of generations) all the parameters of the Kobayashi-Maskawa (K-M) mixing matrix in terms of the quark masses alone. This model which consists of combining the Fritzsch[6] and Stech[7] Ansätzae (two reasonable proposals in the literature which had previously seemed to be mutually exclusive) is in agreement with our present knowledge of the quark masses and mixings. In particular it requires the not yet discovered top quark mass (physical value) to lie in the range

$$25 \text{ GeV} < m_{top} < 45 \text{ GeV}. \qquad (1.1)$$

It seems natural to try to extend this model in some way to the leptons. One can postulate[4] that the charged lepton and neutrino mass matrices have an analogous structure to the two quark mass matrices or alternatively one can embed the leptons as well as the quarks in a grand unified theory (GUT) of some type. Considering the present state of flux of GUT's we may expect the results for leptons obtainable by the latter approach to be more tentative than the results for hadrons. Nevertheless this seems to be an attractive way to attack the problem. The most standard GUT in which to discuss leptons is the one based on the SO(10) group. This has the interesting feature that it

leads to non-zero (Majorana) neutrino masses and a leptonic analog of the K-M matrix. Actually the combined Fritzsch-Stech model has recently been extended to leptons in an SO(10) framework by A. Bottino, C.W. Kim, H. Nishiura and W.K. Sze.[5] Here we will study this model further. We will consider some alternative assumptions for the neutrino mass matrix and will point out a very interesting new possible structure for neutrinos in SO(10). In addition we shall perform a more detailed numerical analysis than in ref. 5. It turns out that there is a sensitivity to the precise values (within the allowed experimental range) of the input parameters.

The strategy is to find enough information from the quark sectors and the charged lepton sector to be able to make predictions for the neutrino sector. In Stech's original Ansätz one has for the up and down quark mass matrices

$$M_u = S$$

$$M_d = \alpha S + A , \qquad (1.2)$$

where S is a symmetric matrix, A is an antisymmetric matrix while α is a real constant. He originally assumed S to be due to the Higgs 10 and A to the Higgs 120. Then one would have for the charged lepton matrix

$$rM_e = \alpha S + \delta A , \tag{1.3}$$

where δ is a real constant and r, very roughly three, is the famous renormalization factor which differentiates between quarks and leptons in their "running" down from the GUT scale to 1 GeV. Now it is easy to convince oneself that, assuming (1.2) in our model, the charged lepton masses cannot be fit with (1.3). A simple solution is to include the Higgs 126; this corresponds to adding $\epsilon S'$, S', $-3S'$ to M_u, M_d, M_e, respectively, where S' is another symmetric matrix. This yields

$$M_d = \alpha M_u + A + (1-\alpha\epsilon)S' , \tag{1.4}$$

which is seen[5] for the case $\alpha\epsilon=1$ to result in the model (1.2) again for the quarks. Given the apparent success of the Stech Ansatz this seems like perhaps the most interesting situation. We shall investigate this case in detail. (It was claimed to be ruled out in ref. 5 because, we believe, they did not allow for the experimental uncertainties in the input parameters.) It turns out that the general situation where $\alpha\epsilon\neq1$ results in very similar physics for the neutrinos as the $\alpha\epsilon=1$ case. A possible advantage of the more general case is that it allows a larger value of top quark mass; (1.1) is replaced by

$$25\ \text{GeV} < m_{top} < 95\ \text{GeV}. \tag{1.5}$$

In the conventional treatment of neutrino masses in SO(10) there are two different contributions to the effective 3x3 (assuming three generations) mass matrix for light neutrinos:

$$M^{eff}_{\nu L} \approx \beta S' - \gamma^{-1} M^T_{\nu D} S'^{-1} M_{\nu D}, \tag{1.6}$$

where $M_{\nu D}$ (see section 4) is roughly proportional to the matrix S, β is a very small number (of the order of 10^{-9}) proportional to the vacuum expectation value of a Higgs $SU(2)_L$ triplet field, and γ is a very large number (of the order 10^{13}) proportional to the vacuum expectation value of a Higgs $SU(2)_L$ singlet. The second term in (1.6) is the see-saw[8] term by which a matrix $\gamma S'$ of very heavy "right-handed" neutrinos feeds in a small mass to the very light "left-handed" neutrinos. It seems fair to say that the relative strength of the two terms, as measured by the values of the numbers β and γ, is a dynamical assumption at our present stage of understanding of GUTs. In ref. 5 it is assumed that the β term is dominant ($SU_L(2)$ triplet dominance). Here we shall also investigate the situation when the second term is assumed to be dominant; this is the ordinary see-saw scenario. In studying this case it is natural to expect that the predictions for neutrino masses and mixings do not vary substantially as the parameters are varied in their allowed ranges. In particular one would not expect a great change in the physical situation if the renormalization parameter r is varied in the range 3 to 4 which may be considered as a conservative uncertainty in this parameter. However that turns out not to be the case in the present model. At a special "magic" value

$$r = \frac{m_b - m_s + m_d}{m_\tau - m_\mu + m_e} \approx 3.0 \text{ to } 3.5 \,, \tag{1.7}$$

the matrix S' becomes singular. Actually the existence of this singularity holds under much more general circumstances wherein one uses the Fritzsch mechanism in SO(10) with any number of $\underline{10}$'s and/or $\underline{120}$'s present. Clearly the approximation in (1.6) is no longer valid at this point and furthermore the dominance of the first term cannot hold either. Instead one must deal with the full 6x6 neutrino matrix

$$M_\nu = \begin{pmatrix} \beta S' & M_{\nu D}^T \\ M_{\nu D} & \gamma S' \end{pmatrix} \,. \tag{1.8}$$

The diagonalization of (1.8) yields, at the singular point, only two superlight neutrinos (scaling like $1/\gamma$), two super heavy neutrinos (scaling like γ) and two degenerate intermediate mass neutrinos which do not scale with γ. Of course by choosing r to be some distance away from the magic value (actually the "magic canyon" is very very narrow for large γ) we can regain the

orthodox see-saw scenario with three superlight neutrinos. However since the magic value of r occurs right around the range predicted from renormalization group arguments[9] it seems interesting to see if we can construct a reasonable theory with two very light neutrinos and one intermediate mass neutrino. Remember that the direct experimental upper bound[10] on the tau neutrinos mass is as large as 50–85 MeV. We will present a simple example in this framework which contains a heavy third neutrino and give a comparison with the many particle physics and astrophysics constraints on neutrino masses and mixings. The simplest model does not quite meet all these criteria and some suggestions are offered to remedy the situation.

For the case when the first term in (1.6) is dominant we see that the proximity of r to its magic value explains the pattern of one almost massless and two nearly degenerate superlight neutrinos observed by Bottino et al.[5]

In section 2 we review the structure of the mass terms in SO(10). Section 3 contains a statement of the assumptions and equations of the model. An analytical treatment of the neutrino masses and mixing matrix at the singular point is given in section 4. In section 5 we give an exact numerical solution of the model in the $\alpha\epsilon=1$ case which corresponds to the original Stech Ansätz holding exactly. We investigate also in this section three distinct scenarios for the neutrinos – the orthodox see-saw, the magic canyon see-saw, and the Higgs $SU(2)_L$

triplet dominance model - and compare with experimental constraints. In section 6 we find the allowed ranges of parameters for the general ($\alpha\epsilon\neq1$) case and show that the neutrino physics is very similar to the $\alpha\epsilon=1$ case in section 5. Finally, in section 7 we make some remarks on the application of our model to the solar neutrino problem.

2. Fermion masses in SO(10)

In SO(10) the quarks and leptons belong to a $\underline{16}$ dimensional spinor representation.[11] Their masses may arise from the usual tree level Higgs' condensation mechanism plus various kinds of radiative corrections. In this section we shall consider the most general tree level structure, taking into account renormalization effects. Of course, this is the initial approach studied by a number of authors when the SO(10) theory was first proposed. Our reasons for returning to it are first that there is now better information on quark masses and K-M parameters and second that there now exist Ansatzae for the quark mass matrices which seem rather promising.

The decomposition $\underline{16}\times\underline{16} = \underline{10}_S+\underline{120}_A+\underline{126}_S$ (S=symmetric, A=antisymmetric) shows that only three types of Higgs fields can contribute at tree level. Our notation for the vacuum values of the relevent Higgs fields is as follows. $v_\pm^{(10)}$ corresponds to

the two $SU(2)_L$ doublets in the $\underline{10}$, where the $\pm$ denotes the third component of $SU(2)_R$ in a labeling of the SO(10) states according to the subgroup $SU(4)\times SU(2)_L\times SU(2)_R$. In $\underline{120}$ there are four $SU(2)_L$ doublets with vacuum values $v_\pm^{(120)}$, $\tilde{v}_\pm^{(120)}$. Finally the $\underline{126}$ contains two $SU(2)_L$ doublets with vacuum values $v_\pm^{(126)}$ as well as an electrically neutral $SU(2)_L$ snglet and an $SU(2)_L$ triplet with respective vacuum values z and w.

Then the mass matrices for the up quarks, down quarks and charged leptons are[12]

$$M_u = v_+^{(10)}S^{(10)}+(v_+^{(120)}+\tilde{v}_+^{(120)}/3)A^{(120)} + v_+^{(126)}S^{(126)}/3$$

$$M_d = v_-^{(10)}S^{(10)}+(-v_-^{(120)}+\tilde{v}_-^{(120)}/3)A^{(120)}-v_-^{(126)}S^{(126)}/3$$

$$rM_e = v_-^{(10)}S^{(10)}-(v_-^{(120)}+\tilde{v}_-^{(120)})A^{(120)}+v_-^{(126)}S^{(126)} , \qquad (2.1)$$

wherein $S^{(10)}$ and $S^{(126)}$ are symmetric matrices of Yukawa coupling constants while $A^{(120)}$ is an anti-symmetric matrix. The factor r is a renormalization factor appropriate to comparing the quark masses with the charged lepton masses at a low energy scale rather than at the grand unified mass scale. Each $\underline{16}$ contains a two component neutrino field belonging to an $SU(2)_L$ doublet as well as an additional $SU(2)_L$ singlet neutrino. Thus we need to consider the composite neutrino matrix

$$M_\nu = \begin{pmatrix} M_{\nu L} & M_{\nu D}^T \\ M_{\nu D} & M_{\nu R} \end{pmatrix} \qquad (2.2)$$

with the 3x3 submatrices,

$$r' M_{\nu D} = v_+^{(10)} S^{(10)} + (v_+^{(120)} - \tilde{v}_+^{(120)}) A^{(120)} - v_+^{(126)} S^{(126)}$$

$$s M_{\nu L} = w S^{(126)}$$

$$s' M_{\nu R} = z S^{(126)} . \qquad (2.3)$$

Here r', s and s' are renormalization constants.

The values for r, r', s, and s' are in principle calculable using renormalization group techniques.. Here we will not be interested in the values of s and s' as they can be absorbed into the parameters w and z in our analysis [see (2.3)]. The precise value of r may, on the other hand, have an important effect on neutrino masses. It is conventionally taken to be very roughly around three since the formula[13] $m_b/m_\tau \approx 3$ is considered one of the indications that grand unification is reasonable. Actually the precise value depends on a number of assumptions both about low energy QCD as well as grand unification. One may see how a value around 3 is achieved by referring to the standard[9] one loop result

$$r(\mu) = \left[\frac{\alpha_3(\mu)}{\alpha(M_G)}\right]^{\frac{4}{11-\frac{2}{3}n_f}} \left[\frac{\alpha_1(\mu)}{\alpha(M_G)}\right]^{\frac{3}{2n_f}} , \qquad (2.4)$$

where n_f is the number of quark flavors, M_G is the grand unification mass scale and μ is the subtraction point. The first factor is due to the difference in SU(3) color charges between quarks and leptons while the second factor is due to the difference of their U(1) hypercharges. (Note that the $U(1)_Y$ factor is of the order of 10% but is less important for comparing M_u and M_d in (2.1)). Using[14] α_s (1 GeV)=0.21 and correspondingly $\alpha(M_G \approx 10^{15}$ GeV$)=.02$ we find from (2.4) that, allowing an extra ten per cent reduction[15] due to finite mass correction factors,

$$r \approx 3.1. \qquad (2.5)$$

Similarly one finds

$$r' \approx 3.5. \qquad (2.6)$$

Other methods[16] of calculation and choices of α_s can yield somewhat larger values of r so one may consider the range $3<r<4$ as reasonable.

Since we will be working with the mass matrices at a scale μ=1 GeV we use in our analysis the values for the fermion running masses evaluated at this scale. We take[17])

$$m_u=5.1 \pm 1.5\ \text{MeV} \qquad m_d=8.9 \pm .26\ \text{MeV} \qquad m_e=.511\ \text{MeV}$$

$$m_c=1.35\pm.05\ \text{GeV} \qquad m_s=0.175\pm0.055\ \text{GeV} \qquad m_\mu=.105\ \text{GeV}$$

$$m_t \geq 35\ \text{GeV} \qquad m_b=5.3 \pm .1\ \text{GeV} \qquad m_\tau=1.784\ \text{GeV} \tag{2.7}$$

We also use the measured KM parameters:

$$|U_{12}| = .225 \pm .010 \qquad \text{[ref. 18]}$$

$$|U_{23}| = .05 \pm .01 \qquad \text{[ref. 19]}. \tag{2.8}$$

We should be careful when quoting quark masses to distinguish the running mass from the "physical" mass. By "physical" mass we will mean the running mass evaluated at the scale of the "physical" mass;

$$m_p = m(m_p). \tag{2.9}$$

In what follows we will always be referring to running masses at 1 GeV unless we explicitly state "physical" mass.

3. Statement of Model

Our starting point is the model[1] in which one simultaneously requires the Stech ansatz[7]

$$M_u = M_u^{\dagger} = S$$

$$M_d = M_d^{\dagger} = \alpha S + A\ , \tag{3.1}$$

where S is a symmetric matrix, A an antisymmetric matrix and α a real constant as well as a Fritzsch-type ansatz[6] for M_u and M_d to hold. This model permits the complete calculation of the K-M matrix in terms of quark mass. Notice that the assumption of hermiticity for the mass matrices is a reasonable one which is imposed primarily for simplicity but also may be derived in certain models. At first one might imagine that the fact that M_d contains an anti-symmetric part is inconsistent with the Fritzsch ansatz which is normally phrased in terms of symmetric matrices which cannot be Hermitean if they are not purely real. However there is no change in the physical predictions of the Fritzsch model when one uses a hermitian matrix of the type

$$M_d = P\begin{pmatrix} 0 & A_d & 0 \\ A_d & 0 & B_d \\ 0 & B_d & C_d \end{pmatrix}P^{-1}, \quad P=\text{diag}(e^{i\beta_1},e^{i\beta_2},e^{i\beta_3}). \tag{3.2}$$

This may, in agreement with (3.1), be decomposed into a symmetric and an antisymmetric piece:

$$M_d = \begin{pmatrix} 0 & A_d\cos\beta_{12} & 0 \\ A_d\cos\beta_{12} & 0 & B_d\cos\beta_{23} \\ 0 & B_d\cos\beta_{23} & C_d \end{pmatrix}$$

$$+i \begin{pmatrix} 0 & A_d\sin\beta_{12} & 0 \\ -A_d\sin\beta_{12} & 0 & B_d\sin\beta_{23} \\ 0 & -B_d\sin\beta_{23} & 0 \end{pmatrix} , \qquad (3.3)$$

where $\beta_{ij}=\beta_i-\beta_j$. We may recall that an important part of Stech's motivation for (3.1) is that it is a reasonable approximation to the situation for simple Higgs schemes in unified models like SO(10) and E(6). Of course the model proposed may hold in other schemes too. To get relationships between the lepton and quark mass matrices it is necessary, however, to adopt a particular grand unified model. Here we will study the case of SO(10) with Higgs mesons of each relevant type present, namely the 10, 120 and 126. The SO(10) group is a standard one for obtaining information about neutrino masses and is also likely to be a physically sensible subgroup for more complicated models such as the superstring model. An interesting new feature in our approach concerns the neutrino masses and mixings. While Bottino et.al.[5] keep only the $M_{\nu L}$ contribution in (2.2) we find that the "see-saw" contribution

may be extremely important. In particular, for a value of the renormalization parameter r in the expected range we find that the "see-saw" mechanism results in two rather than three light Majorana neutrinos. The third neutrino (mostly the τ neutrino) is close to a Dirac neutrino with mass roughly of the order of one hundred MeV. By adjustment of parameters the third neutrino mass can be tuned continuously down to the usual superlight scale. The possibility of a heavy tau neutrino is very interesting because the present experimental bound[10] on its mass is roughly of the order of one hundred MeV. We remark that this result holds with a much more general set of Higgs multiplets than the ones to be considered here. Whether or not the tau neutrino is truly heavy, the present model shows that the see-saw mechanism in SO(10) does not automatically give three light neutrinos.

We now rewrite[5] (2.1) with the additional assumption that the $A^{(120)}$ piece in M_u is negligible. (The success of the previous model shows this to be reasonable.)

$$M_u = S + \epsilon S'$$

$$M_d = \alpha M_u + A + (1-\alpha\epsilon)S'$$

$$rM_e = \alpha M_u + \delta A - (3+\alpha\epsilon)S' . \qquad (3.4)$$

Here $S = v_+^{(10)} S^{(10)}$, $S' = -1/3 v_-^{(126)} S^{(126)}$ and $A = (-v_-^{(120)} + \tilde{v}_-^{(120)}/3) A^{(120)}$. α, δ and ϵ are real constants. S and S' are real symmetric matrices while A is a pure imaginary anti-symmetric matrix. The model is completed by requiring M_d to have the form (3.2) and to furthermore require

$$M_u = \begin{pmatrix} 0 & A_u & 0 \\ A_u & 0 & B_u \\ 0 & B_u & C_u \end{pmatrix}$$

$$rM_e = Q \begin{pmatrix} 0 & A_e & 0 \\ A_e & 0 & B_e \\ 0 & B_e & C_e \end{pmatrix} Q^{-1}, \quad Q = \mathrm{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}) \tag{3.5}$$

The parameters of the charged lepton Fritzsch matrix for example, are related to masses by[6]

$$A_e = \sqrt{\frac{m_e m_\mu m_\tau}{C_e}}$$

$$B_e = C_e^{-1/2} \sqrt{(m_\tau + m_e)(m_\tau - m_\mu)(m_\mu - m_e)}$$

$$C_e = m_\tau - m_\mu + m_e. \tag{3.6}$$

Note that we are using the exact rather than the approximate forms of A_e, B_e and C_e etc. Finally, using (3.2) and (3.5) in (3.4) we find[5]

$$r(1-\alpha\epsilon)A_e \cos\alpha_{12} = 4\alpha A_u - (3+\alpha\epsilon)A_d \cos\beta_{12}$$

$$r(1-\alpha\epsilon)B_e \cos\alpha_{23} = 4\alpha B_u - (3+\alpha\epsilon)B_d \cos\beta_{23}$$

$$rA_e \sin\alpha_{12} = \delta A_d \sin\beta_{12}$$

$$rB_e \sin\alpha_{23} = \delta B_d \sin\beta_{23}$$

$$r(1-\alpha\epsilon)C_e = 4\alpha C_u - (3+\alpha\epsilon)C_d, \tag{3.7}$$

where $\alpha_{ij} = \alpha_i - \alpha_j$. Assuming the quark and lepton masses are known, (3.7) comprises a set of 5 equations for the seven quantities α_{12}, α_{23}, β_{12}, β_{23}, α, δ and ϵ. If we agree to furnish the two known Kobayashi Maskawa matrix elements $|U_{us}|$ and $|U_{cb}|$ in addition to the masses we can determine all parameters of the charged particle sector of the model except for the case $\alpha\epsilon=1$ to be discussed below. This is non-trivial not only because we can go on to predict lepton mixing angles and neutrino masses but also because self-consistency of the equations forces important restrictions on the allowed ranges of parameters.

Notice that for the choice

$$\alpha\epsilon = 1, \tag{3.8}$$

the first two of eqs. (3.4) reduce to the form of (3.1). This (original[1]) model appears to be close to experiment, predicting a top quark mass in the 25-45 GeV region. Allowing $\alpha\epsilon$ to differ from unity would enable one to accomodate a somewhat higher value of the t-quark mass. In the $\alpha\epsilon=1$ case the counting of independent parameters changes. Specifying all the quark masses will yield[1] β_{12}, β_{23} and α. The third of (3.4) can now be seen to yield two equations for the three quantities α_{12}, α_{23} and δ. Thus, for the very interesting case $\alpha\epsilon=1$, there is the convenient feature that, while the quark mixing matrix is already predicted, the charged lepton matrix may be adjusted via a free parameter which we may take to be δ. Finally one recovers the original equations[1]

$$\cos\beta_{12} = \alpha \frac{A_u}{A_d}$$

$$\cos\beta_{23} = \alpha \frac{B_u}{B_d}$$

$$\alpha = \frac{C_d}{C_u} \qquad (3.9)$$

for $\alpha\epsilon=1$ and in addition the third and fourth equations of (3.7).

4. Neutrino masses[20]

First we rewrite the submatrices of (2.3) with the notation of (3.4):

$$M_{\nu D} \approx S/r'$$

$$M_{\nu L} = -3wS'/(sv_-^{(120)}) \equiv \beta S'$$

$$M_{\nu R} = -3zS'/(s'v_-^{(126)}) \equiv \gamma S'. \qquad (4.1)$$

In the first of these equations we have approximated the first of (2.3) by including only the $S^{(10)}$ term. This is reasonable numerically and is in any case irrelevant for an important point we wish to make. Notice that the vacuum value z for the $SU(2)_L$ singlet in the 126 is expected to be very very large, perhaps equal to the grand unified scale while the vacuum value w for the $SU(2)_L$ triplet in the 126 must be extremely small to avoid altering the predicted mass ratio of the neutral and charged intermediate vector bosons. Ordinarily one is primarily concerned with the effective 3x3 light neutrino mass matrix obtained by bringing (2.2) to block diagonal form and retaining the upper left block:

$$M^{eff}_{\nu L} \approx M_{\nu L} - M^T_{\nu D} M^{-1}_{\nu R} M_{\nu D} \ . \qquad (4.2)$$

The second term in (4.2) is the see-saw term; we shall initially neglect the first term.

An unusual point of interest is that the matrix S' and hence, by (4.1), the matrix $M_{\nu R}$ may become singular. This may be seen in the following way. From (3.4) we have

$$\mathrm{Tr}S' = S'_{33} = (\mathrm{Tr}M_d - r\,\mathrm{Tr}M_e)/4 = [(m_b - m_s + m_d) - r(m_\tau - m_\mu + m_e)]/4, \qquad (4.3)$$

where the Fritzsch ansatz was assumed. Since $\det S' = -S'_{33}(S'_{12})^2$ we see that S' becomes singular when

$$r = \frac{m_b - m_s + m_d}{m_\tau - m_\mu + m_e} = 3.0 \text{ to } 3.2 \ , \qquad (4.4)$$

where the uncertainty corresponds to the theoretical mass uncertainties quoted in (2.7). Actually the range[21] 3.0 to 3.5 might be a more conservative choice in (4.4). This "magic" value of r should be compared with (2.5). We would like to point out that eqs. (4.3) and (4.4) will hold without approximation when any number of Higgs 10's and Higgs 120's are present if the Fritzsch ansatz is assumed. (However, only a single 126 is allowed unless all 126 contributions are proportional to each other.) One may always choose r to avoid the singular point but it then seems disconcerting that a deep canyon lies very close. It is even conceivable that a "fixed-

point" type of argument similar to those proposed for the quark sector[22] might be developed which would attract the system to its singular point. In any event there would be some residual effects (stronger for smaller z) associated with the system necessarily being close to the singular point.

We have performed a numerical analysis of this system but before presenting it, we shall give an analytic discussion of the singular situation. Naively (4.2) indicates an infinite mass neutrino when $M_{\nu R}$ is singular. This is due to the approximation involved in its derivation. In the singular case we must diagonalize the full 6x6 neutrino mass matrix. Specializing to the $\beta=0$ case we then write

$$M_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & a & 0 & b \\ 0 & 0 & 0 & 0 & b & c \\ 0 & a & 0 & 0 & a' & 0 \\ a & 0 & b & a' & 0 & b' \\ 0 & b & c & 0 & b' & c' \end{pmatrix} , \qquad (4.5)$$

where $a \approx S_{12}/r'$, $b \approx S_{23}/r'$, $c \approx S_{33}/r'$, $a' = \gamma S'_{12}$, $b' = \gamma S'_{23}$, $c' = \gamma S'_{33}$. Since $\det M_\nu = -a^4 c^2$, we see that the singularity of $M_{\nu R}$ obtained by imposing $c'=0$ does not result in any zero eigenvectors for the full matrix M_ν. We can understand what is happening for $c'=0$ by successively first diagonalizing the lower right 3x3 submatrix and then the upper left 4x4 submatrix. In the new basis M_ν has the form

$$M'_\nu = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & \sqrt{(x^2+a'^2c^2)/(a'^2+b'^2)} & & & \\ & & & -\sqrt{(x^2+a'^2c^2)/(a'^2+b'^2)} & & \\ & & & & \sqrt{a'^2+b'^2} & \\ & & & & & -\sqrt{a'^2+b'^2} \end{pmatrix} +$$

$$+ \text{(correction terms)},$$

$$x \equiv ab' - ba' \quad . \tag{4.6}$$

The correction terms are of order a and will connect the first four indices with the last two. They will, in perturbation theory, give mass (of order a^2/a') to the first two neutrinos but not have an appreciable effect on the other eigenvalues. The most striking feature of (4.6) is that there are only two light neutrinos with masses of order a^2/a' and only two super heavy neutrinos with masses of order a'. The other neutrinos both have the same mass (a sign is irrelevant of course) of order a. In fact these two 2-component neutrinos can be regarded as a single Dirac neutrino. This is the neutrino which is presumably mostly associated with the τ-lepton. Note that its mass m_3 is independent of the super large vacuum expectation value z and is given by

$$m_3^2 = \frac{x^2+c^2a'^2}{a'^2+b'^2} \quad . \tag{4.7}$$

This expression is roughly minimized by setting $a'=0$ which yields

$$m_3 \approx a \approx \frac{1}{r'} \sqrt{m_u m_c} \approx 24 \text{ MeV} , \tag{4.8}$$

where we used $S_{12}=(M_u)_{12}$ and $r'=3.5$ was taken as discussed in section 2. Of course by detuning r slighly from its magic value m_3 may be continuously driven down to a superlight value of order a^2/a'. Actually, at magic r, the minimum value (4.8) of m_3 corresponds to an unphysical situation as can be seen from the approximate expressions for the third and fourth eigenvectors of (4.5):

$$\psi_{3,4} \approx \left|\frac{1}{2(x^2+a'^2c^2)}\right|^{1/2} \begin{pmatrix} 0 \\ x \\ -ca' \\ \pm\sqrt{x^2+a'^2c^2} \\ 0 \\ 0 \end{pmatrix} \quad . \tag{4.9}$$

We see that for $a'=0$ the heavy (but not superheavy) eigenvectors have a substantial component along the mu rather than the tau direction. This is modified (but not by very much in our model) by the need to diagonalize the charged lepton matrix M_e.

Effectively, these two neutrinos are behaving like a Dirac mu neutrino of mass 24 MeV which is ruled out by experiment.[23] A conventional superlight mu neutrino is obtained by considering the x=0 case. Then

$$m_3 \approx \frac{ca'}{r'/b'} \approx \frac{ca}{r'/b} \approx \frac{1}{r'} \sqrt{m_u m_t} \approx 147 \text{ MeV}.$$

This value is a bit larger than the present experimental bound which requires[10]

$$m(\nu_\tau) < 85 \text{ MeV}. \qquad (4.10)$$

Thus some deviation from magic r is indicated to achieve a realistic theory.

More detailed information about the light neutrinos may be obtained from a study of the secular equation of (4.5):

$$\lambda^6 + \sum_{n=0}^{5} d_n \lambda^n = 0 ,$$

$$d_0 = -a^4 c^2,$$

$$d_1 = 2abcx - c'a^2(a^2+b^2),$$

$$d_2 = (a^2+b^2)^2 + x^2 + c^2(a'^2+2a^2),$$

$$d_3 = -2bcb' + c'(a'^2+2a^2+b^2),$$

$$d_4 = -2a^2-2b^2-c^2-a'^2-b'^2,$$

$$d_5 = -c' . \qquad (4.11)$$

Retaining only the leading terms for the roots λ of order a^2/a' yields a cubic equation

$$\lambda^3(c'a'^2)+\lambda^2(x^2+a'^2c^2)+\lambda[2abcx-c'a^2(a^2+b^2)]-a^4c^2 = 0 . \qquad (4.12)$$

Thus, in general, one finds three light neutrinos. But when $c'=0$, (4.12) collapses to a quadratic and we find the two light neutrino masses:

$$\lambda_{1,2} = \frac{xac}{a'^2c^2+x^2} [-b\pm \sqrt{b^2+a^2(1+ \frac{a'^2c^2}{x^2})}] . \qquad (4.13)$$

The two super-heavy neutrino masses are easily seen from (4.11) to be $\lambda_{5,6} = \pm \sqrt{a'^2+b'^2}$ in agreement with (4.6). Finally using this with (4.13) in the formula $\pi\lambda_i = -a^4c^2$ gives $\lambda_{3,4} = \pm\sqrt{(x^2+a'^2c^2)/(a'^2+b'^2)}$ as in (4.7). Knowing the λ_i we may find the six components of the eigenvectors $X(\lambda_i)$ of M_ν by directly substituting into the eigenvalue equations. We have the following exact formulas in the $c'=0$ case:

$$\frac{x_2(\lambda)}{x_1(\lambda)} = \frac{b\lambda^2}{b^2\lambda^2-(a^2-\lambda^2)(c^2-\lambda^2)} \left[-\frac{\lambda}{a}(b'+\frac{b\lambda}{c})+\frac{1}{b\lambda}(\lambda^2-c^2)(\frac{b^2\lambda}{ac}-a')\right],$$

$$\frac{x_3(\lambda)}{x_1(\lambda)}=[b^2\lambda^2-(a^2-\lambda^2)(c^2-\lambda^2)]^{-1}\left[\frac{-\lambda c}{a}(\lambda^2-a^2)(b'+\frac{b\lambda}{c})+bc\lambda(\frac{b^2\lambda}{ac}-a')\right],$$

$$\frac{x_4(\lambda)}{x_1(\lambda)} = \frac{\lambda}{a}\frac{x_2(\lambda)}{x_1(\lambda)} - \frac{b\lambda}{ac}\frac{x_3(\lambda)}{x_1(\lambda)} + \frac{b^2\lambda}{ac},$$

$$\frac{x_5(\lambda)}{x_1(\lambda)} = \frac{\lambda}{a},$$

$$\frac{x_6(\lambda)}{x_1(\lambda)} = \frac{\lambda}{c}\frac{x_3(\lambda)}{x_1(\lambda)} - \frac{b\lambda}{ac}. \qquad (4.14)$$

For the <u>light</u> neutrino eigenvectors we have the approximations

$$\frac{x_2(\lambda)}{x_1(\lambda)} \approx -\lambda a'/a^2, \qquad a'\neq 0$$

$$\frac{x_3(\lambda)}{x_1(\lambda)} \approx \frac{-\lambda x}{a^2 c}. \qquad (4.15)$$

It is amusing that in the interesting $x=0$ case discussed above the two light neutrinos are also degenerate; then (4.13) shows that $\lambda_{1,2}\to\pm a^2/a'$. Let us use this special case to illustrate our conventions for the lepton mixing matrix. If M_ν is brought to diagonal form by the transformation $V^T M_\nu V$=diagonal the mass eigenstate neutrino fields ν_i are related to the original neutrino fields ρ_i by

$$\rho_L = V\nu_L, \quad V = (\vec{X}(\lambda_1) \ldots \vec{X}(\lambda_6)). \qquad (4.16)$$

The charged lepton mass eigenstate fields e_L are related to the original fields E_L by

$$E_L=\Omega e_L \qquad (4.17)$$

where[24] in our model

$$\Omega^\dagger \approx \begin{pmatrix} 1 & -\sqrt{\frac{m_e}{m_\mu}}e^{i\alpha_{12}} & -\sqrt{\frac{m_e}{m_\tau}}e^{i\alpha_{13}} \\ \sqrt{\frac{m_e}{m_\mu}}e^{-i\alpha_{12}} & 1 & \sqrt{\frac{m_\mu}{m_\tau}}e^{i\alpha_{23}} \\ \frac{m_\mu}{m_\tau}\sqrt{\frac{m_e}{m_\tau}}e^{-i\alpha_{13}} & -\sqrt{\frac{m_\mu}{m_\tau}}e^{-i\alpha_{23}} & 1 \end{pmatrix}, \qquad (4.18)$$

and a change of phases has been made to keep the diagonal elements equal to +1. Finally the analog of the KM matrix for leptons is the 3x6 rectangular matrix

$$K_{ij} = \sum_{k=1}^{3} (\Omega^\dagger)_{ik} V_{kj}. \qquad (4.19)$$

The lepton charged current weak interaction term is

$$\mathcal{L} = \frac{ig}{\sqrt{2}} W_\mu^- \bar{e}_L \gamma_\mu K \nu_L + h.c. \qquad (4.20)$$

in a standard notation. A discussion of the parametrization of K exists in the literature.[25] In the x=0 case

$$\nu \approx \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 & \\ -1 & 1 & 0 & 0 & \\ 0 & 0 & -1 & -1 & \ldots \\ 0 & 0 & 1 & -1 & \\ & \vdots & & & \end{pmatrix} , \qquad (4.21)$$

as we may easily verify using (4.16), (4.15) and (4.9). The neutrino associated with the tau lepton is

$$\nu_\tau \approx k_{31}\nu_1 + k_{32}\nu_2 - \frac{1}{\sqrt{2}} (\nu_3+\nu_4) .$$

We redefine $\nu_3 = 1/\sqrt{2}\ (\nu_3+\nu_4)$ and notice that it may be considered to be the upper two components of a four component Dirac spinor in a γ_5 diagonal representation of the Dirac matrices. The orthogonal component $\nu_4 = 1/\sqrt{2}\ (\nu_3-\nu_4)$ is non-interacting. Thus for degenerate third and fourth neutrinos we may deal with an effective 3x3 matrix V_{eff} for the light neutrino interactions. For the case of (4.21) we have

$$V_{eff} \approx \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\,-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{pmatrix} . \qquad (4.22)$$

Actually in this special case there are three degenerate sets of neutrinos. For the lightest ones, however, both components are involved in charged current weak interactions.

Because we have the freedom to move the system to the singular point it should be clear that a number of exotic neutrino mass and mixing models may be constructed. These will be discussed in the remainder of this paper.

5. Exact Stech-Fritzsch model in SO(10)

Here we consider the $\alpha\epsilon=1$ case. If one precisely knew the 6 quark masses one could predict the 4 independent parameters of the K-M matrix. Since the t-quark mass is not really known and since there is a fair amount of uncertainty in the value of the strange quark mass m_s we shall instead take the six quantities $m_u/m_d = .57$, $m_d/m_s = .056$, $m_c = 1.35$ GeV, $m_b = 5.3$ GeV, $|U_{12}|=.225\pm.010$, $|U_{23}|=.05\pm.01$ as input and predict m_s and m_t. The results are shown in Fig. 1 where the white area is the allowed region in the m_t-m_s plane. The width of this region is due to the uncertainties in $|U_{12}|$ and $|U_{23}|$. Note that the values of m_s and m_t are the "running" values at a scale of 1 GeV. The maximum allowed physical value of m_t is around 45 GeV. Choosing different values for the ratio m_d/m_s will change the value of $|U_{12}|$ and will generally reduce the allowed range in Fig. 1.[26)]

In this ($\alpha\epsilon=1$) case we have now determined the quark parameters α, β_{12} and β_{23} in terms of quark masses (see (3.9)). The charged lepton parameters α_{12}, α_{23} and δ are constrained by the two equations

$$\sin\alpha_{12} = \delta A_d \sin\beta_{12}/rA_e,$$

$$\sin\alpha_{23} = \delta B_d \sin\beta_{23}/rB_e , \qquad (5.1)$$

extracted from (3.7). We will regard δ as a free parameter which however is bounded as $|\delta| < rA_e/A_d|\sin\beta_{12}|$, typically around 0.6. Given a choice of δ, the remaining parameters are determinable from (5.1). The antisymmetric matrix A may be read off from (3.3) while S and S′ may now be found from (3.4). In particular

$$S'_{12} = (\alpha A_u - rA_e \cos\alpha_{12})/4$$

$$S'_{23} = (\alpha B_u - rB_e \cos\alpha_{23})/4$$

$$S'_{33} = (\alpha C_u - rC_e)/4 , \qquad (5.2)$$

while $S=M_u-S'/\alpha$. There are, of course, no predictions for the charged lepton masses but we can go on to make predictions for the neutrino masses and lepton mixing angles as discussed in section 4.

To make an exploration of the lepton system we shall adopt the following specific mass choices in the quark system:

$$m_s = .160 \text{ GeV}, \qquad m_t = 53.0 \text{ GeV}. \qquad (5.3)$$

This results in

$$|U_{12}| = 0.224 \quad , \quad |U_{23}| = 0.0557,$$

$$\alpha = 0.0997 , \quad \beta_{12} = 77.4° , \quad \beta_{23} = 19.8° , \qquad (5.4)$$

where we have, in addition, assumed that β_{12} and β_{23} are both in the first quadrant.[27] The neutrino masses and mixings have an interesting dependence on the quantities δ and r. To simplify things we shall first set the vacuum value of the $SU(2)_L$ Higgs triplet measured by β (see eq.(4.1)) to zero. This is the pure see-saw case. The dependence on r' of computed quantities is shown in section 4 to be trivial and we shall take $r'=3.5$ (see eq.(2.6)). The dimensionless quantity γ, defined in (4.1), is naturally expected to be roughly around the ratio $M_G/M_W \approx 10^{13}$. We have seen in the last section that when r approaches very very closely to its magic value given in (4.4), the neutrino mass spectrum will be dramatically distorted. There is also another unusual region. We noted that when $a'=0$ (see (4.5)) the roles of the second and third neutrinos appeared to be reversed when r was magic. Actually for any r there will be a drastic distortion of the spectrum when $a'=0$ (Eq. (4.12) for the light neutrino masses then becomes a quadratic rather than a cubic). The condition $a'=0$ translates to the condition on δ:

$$\delta^2 = \frac{r^2 A_e^2 - \alpha^2 A_u^2}{A_d^2 - \alpha^2 A_u^2} \quad . \tag{5.5}$$

For the choice (5.3) this is well approximated by $|\delta| \approx 0.19r$ for r near 3. Finally another special case discussed for magic r in

the last section, $X \equiv ab' - ba' = 0$, does not distort the mass spectrum drastically but does result in similar masses for the two lightest neutrinos (Konopinski-Mahmoud[28] or pseudo-Dirac case[29]). This corresponds to

$$\delta^2 = r^2 \left[\left(\frac{B_e}{B_u}\right)^2 - \left(\frac{A_e}{A_u}\right)^2 \right] \left[\left(\frac{B_d}{B_u}\right)^2 - \left(\frac{A_d}{A_u}\right)^2 \right]^{-1} , \tag{5.6}$$

which amounts to $|\delta| \approx 0.17r$ for the choice (5.3).

If r is not very close to (4.4) and if $|\delta|$ is less than about 0.4, the results are not terribly sensitive to their exact values. In this case there will be just three light neutrinos, as usual. For example with the choices $r=3.00$ and $\delta=0.200$ we have three light neutrinos and three superheavy neutrinos:

$$m_1 = 1.94\times10^{-5} \text{ eV} , \quad m_2 = 0.0059 \text{ eV},$$

$$m_3 = 7.90 \text{ eV} , \quad m_4 = 2.29\times10^{7} \text{ GeV},$$

$$m_5 = 1.00\times10^{11} \text{ GeV}, \quad m_6 = 1.28\times10^{11} \text{ GeV}. \tag{5.7}$$

In obtaining (5.7) we assumed $\gamma=10^{12}$ which gives, for the sake of illustration a "muon" neutrino mass m_2 of the correct size to solve the solar neutrino problem on a recent interpretation.[30] For any other choice of γ we should multiply m_1, m_2 and m_3 by

(10^{12} GeV/γ) and m_4, m_5 and m_6 by ($\gamma/10^{12}$ GeV). The 3x6 lepton mixing matrix K, defined in (4.19) is now just a 3x3 matrix to a good approximation. With phase changes on the neutrino fields to make all the masses positive we find

$$K = \begin{pmatrix} 1.00 & .0240@161° & .010@-66° \\ .0247@21° & 0.99 & .133@185° \\ .0086@-97° & .133@-5° & 0.99 \end{pmatrix}, \qquad (5.8)$$

where $\rho@\theta \equiv \rho e^{i\theta}$. The pattern of masses and mixings given by (5.7) and (5.8) is really the kind of thing one would expect from the see-saw mechanism in a grand unified theory.

Interestingly the mixing angle $|K_{12}|$ is not very much larger than the minimum allowed[31] for solution of the solar neutrino problem. There is no important dependence on r away from the magic canyon, but the CP violating phases and mixing angles will depend on δ. Notice that there are[25] (because we are dealing with massive two component neutrinos) three independent CP violation phases in (5.8) which may be approximately taken as arg K_{12}, arg K_{13} and arg K_{23}. For lepton number conserving processes the relevant combination is an <u>invariant</u> phase[32] $\approx$ arg K_{12} + arg K_{23} − arg K_{13} = 52°.

When δ=0 there is no leptonic CP violation. In that case the neutrino masses are about the same as (5.7) but the mixing matrix becomes

$$K = \begin{pmatrix} 1.00 & .0127@90° & .0099@-90° \\ .0138@90° & 0.99 & .128@180° \\ .0082@-90° & .128 & 0.99 \end{pmatrix}. \qquad (5.9)$$

Observe that the invariant phase vanishes in (5.9) and furthermore that a 90° phase as well as a zero phase corresponds to no CP violation for the lepton number changing processes.[33]

As δ increases towards the special value (5.6) the two light neutrino masses approach each other and the mixing matrix shifts accordingly. For example when δ=0.50 we find $m_1=5.34\times10^{-4}$ eV, $m_2=8.14\times10^{-4}$ eV, $m_3=7.81$ eV, $m_4=2.79\times10^{6}$ GeV, $m_5=9.82\times10^{10}$ GeV, $m_6=1.26\times10^{11}$ GeV and

$$K = \begin{pmatrix} 0.75 & 0.66@96° & .0018@-115° \\ 0.64@83° & 0.73 & 0.24@94° \\ 0.15@-187° & 0.19@83° & 0.97 \end{pmatrix}. \qquad (5.10)$$

This is a "pseudo-Dirac"[20] model in which there is a large cancellation in the contribution to neutrinoless double beta decay: $(K_{11})^2 m_1 + (K_{12})^2 m_2$. Finally, if δ is raised a little further to reach the special value (5.5), there will be only two light neutrinos as in the case discussed in section 4. This occurs even if r does not take its magic value. However the second ("muon") neutrino will then be the heavy one, an obviously undesirable situation.

The models above given in (5.7)-(5.10) will yield neutrino masses and mixings which don't conflict with any experimental or astrophysical requirements, apart from the IIEP[34] and

Simpson[35] measurements. Thus they may be considered as reasonable, orthodox predictions of the model. However, as we have seen, this model can predict very different kinds of results if r is in the magic canyon. The fact that the magic canyon occurs right around the expected range for r makes it worthwhile to see if a credible model can be constructed in this case. With the mass choices (5.3) we find the magic value of r to be $r \approx 3.065707$. For simplicity take $\delta=0$, which eliminates CP violation but will not (for $|\delta|<0.4$) have a large effect on the pattern of masses and mixings. With the value[36] $\gamma=10^{10}$ we have for magic r the masses

$$m_1 = .0018 \text{ eV}, \quad m_2 = .56 \text{ eV}$$

$$m_3 = m_4 = .46 \text{ GeV}$$

$$m_5 = m_6 = 1.21\times10^9 \text{ GeV}, \tag{5.11}$$

in agreement with the pattern of section 4. Note that for any other γ choice, m_1 and m_2 should be multipled by $(10^{10}\text{ GeV}/\gamma)$, m_5 and m_6 by $(\gamma/10^{10}\text{ GeV})$ while m_3 and m_4 are unchanged (at magic r). The lepton mixing matrix K of (4.19) is now well approximated by a 3x4 matrix:

$$K = \begin{pmatrix} 1.00 & .012i & .0070i & -.0070 \\ .014i & .99 & .090 & .090i \\ .0082i & -.13 & .70 & .70i \end{pmatrix}. \tag{5.12}$$

(This may be actually reduced to an effective 3x3 matrix as discussed in (4.22).) Clearly this value of $m_3=m_4$ contradicts the experimental[10] indications that $m_\tau<50\text{-}85$ MeV.

We saw in the last section that it is difficult to achieve a realistic theory by sticking to magic r and varying δ. Thus we now (keeping $\delta=0$ and $\gamma=10^{10}$) allow r to deviate slightly from its magic value. Fine tuning r to 3.065700 yields the same super light masses m_1 and m_2 and the same superheavy masses m_5 and m_6 as in (5.11) but now the degeneracy of m_3 and m_4 is split and we have

$$m_3 = 7.4 \text{ MeV}, \quad m_4 = 28 \text{ GeV}. \tag{5.13}$$

If one were to change γ, m_3 would now scale like $1/\gamma$ and m_4 would scale like γ. The mixing matrix is now:

$$K = \begin{pmatrix} 1.00 & .012i & 0.0098i & 1.6\times10^{-4} \\ 0.014i & .99 & 0.13 & -2.1\times10^{-3}i \\ 0.0082i & -.13 & .99 & 0.016i \end{pmatrix}, \tag{5.14}$$

which shows on comparison with (5.12) that the fourth neutrino is decoupling from the first three. The effect on K of changing γ is essentially to multiply the fourth column by $(10^{10}\text{ GeV}/\gamma)$.

To get a feeling for the width of the magic canyon, a decrease of r by one part in 10^5 (say to $r=3.065670$) will reduce m_3 to 1.4 MeV and increase m_4 to 147 GeV, still keeping $\gamma=10^{10}$. (Choosing $\gamma=10^{12}$ instead would yield a 14 keV "τ neutrino".)

Can the model described by (5.13) and (5.14), which seems typical of the unusual situation associated with the existence of the magic canyon, be made compatible with the many experimental constraints on neutrino masses and mixings? We shall concentrate on the value of m_3 and the appropriate matrix elements of K. Bounds on some of these matrix elements have been reported based[37)] on the searches for $\pi\to\mu\nu_3$ [38)] and $\pi\to e\nu_3$.[39)] If $m(\nu_3)$ is around 7 MeV the former experiment suggests $|K_{23}|$ should be less than about 0.14 to 0.17 which seems just consistent with (5.14). There is a smaller bound from the $\pi\to e\nu_3$ experiment for $|K_{13}|$ which is in the region of 0.01. This seems again just consistent with (5.14). There are also astrophysical constraints[40)] which require that a neutrino of about 10 MeV decay with a lifetime less than about 10^4 sec. Our ν_3 should have a dominant decay mode $\nu_3\to e^-e^+\nu_e$ with lifetime about

$$\tau(\nu_3\to e^-e^+\nu_e) \approx \left|\frac{m(\mu)}{m(\nu_3)}\right|^5 \frac{\tau(\mu\to e\bar{\nu}_e\nu_\mu)}{|K_{13}|^2} . \qquad (5.15)$$

This results in $|K_{13}|\gtrsim.005$ which is again consistent with

(5.14). On the other hand, this model does not seem to be within the bounds obtained from the non-observation of neutrinoless double beta decay. That process results in the rough requirement[41)]

$$\left|\sum_{i=1}^{3} m_i (K_{1i})^2\right| \approx m_3|(K_{13})^2| \lesssim 10 \text{ eV} , \qquad (5.16)$$

which would require $|K_{13}|$ to be less than about .001. One might think that a possible way out would be to lower m_3 to around 10-100 keV. But then there might be a difficulty with astrophysical bounds since the quick decay mode into $e^+e^-\nu_e$ would no longer be allowable energetically. The "ordinary" decay modes into 3ν and $\nu\gamma$ are expected to be rather slow. If a very light scalar particle, J associated with spontaneous breakdown of a family-type lepton number symmetry existed this difficulty might however be overcome.[42)] Thus although the most straightforward attempt to satisfy experimental constraints for a model with a relatively heavy third neutrino does not quite work, it is possible that the mechanism discussed here may eventually prove useful in a viable physical theory.

Another way to try to reduce $|K_{13}|^2 m_3$ might involve reducing $|K_{13}|$ and allowing m_3 to be as large as about 30 MeV. Here it may be relevant to point out that we are using the approximation in (4.1) that $M_{\nu D}\propto S$; the neglected terms in (2.3) proportional to A and S' could play a role. Furthermore the assumption that the $SU(2)_L$ triplet vacuum value (essentially β in (4.1)) vanishes might not be realistic. The first term in (4.2) could then interfere with the second see-saw term. However we have found that this interference does not seem to reduce $|K_{13}|$.

Perhaps the most elegant way to obtain a realistic model with a heavy ν_τ would be to work at magic r and find an $M_{\nu D}$ giving $m(\nu_\tau)$ about 10 MeV as well as a matrix K similar to (5.12). Then, because ν_τ would be of Dirac type there would be no contribution to neutrinoless double beta decay. If $m(\nu_\tau)$ is eventually measured to be large, a theory of this kind would be required.

The case when $M_{\nu L}$ in (4.2) is actually assumed to be the dominant one is also of interest. Then $M^{eff} \approx M_{\nu L} = \beta S'$ and we might expect some unusual features because S' is close to being a singular matrix ($S'_{33} \approx 0$). For the $M_{\nu L}$ term the effects of the magic canyon are very important for any realistic r in contrast to the see-saw term where the canyon's width is drastically reduced by the factor γ. Roughly speaking, the neutrino mass matrix will look like

$$M_\nu^{eff} \approx \beta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & S'_{23} \\ 0 & S'_{23} & 0 \end{bmatrix} , \qquad (5.17)$$

which has eigenvalues 0, $\pm\beta S'_{23}$. This general pattern persists

for the entire range of r and δ in that one has a very light first neutrino and two roughly degenerate heavier neutrinos. The amount of CP violation depends on the choice of δ. For example with r=3.00 and δ=0.200 we find

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = (10^9\beta) \begin{bmatrix} 2.2\times10^{-5} \\ 0.10 \\ 0.13 \end{bmatrix} \text{eV} , \qquad (5.18)$$

and

$$K = \begin{bmatrix} 1.00 & .042@-62° & .062@-169° \\ .070@-110° & .82 & .57@-91° \\ .029@1° & .57@-89° & .82 \end{bmatrix} . \qquad (5.19)$$

The characteristic difference between this model and the orthodox see-saw model [see (5.7) and (5.8) for example] is that in the latter the third neutrino mass is considerably heavier than the second which is considerably heavier than the first. Notice that in the model of (5.18) and (5.19) there is a very strong bound on β from $\nu_\mu - \nu_\tau$ oscillation experiments. Eq. (5.19) shows that the ν_μ-ν_τ mixing is just about maximal for oscillations. Consulting the results of the E531-FNAL experiment[43)] shows that $(m_3)^2-(m_2^2) \lesssim 5$ eV^2 which yields

$$\beta \lesssim 2.5\times10^{-8} . \qquad (5.19)$$

In such a model the heaviest neutrino should be less than about 3 eV.

6. General case where $\alpha\epsilon\neq1$

In the last section we discussed the neutrino masses and mixings for the $\alpha\epsilon=1$ case where the original Stech ansatz (3.1) holds exactly. That model illustrates the fact that (3.1) can hold even when a 126 Higgs is present and leptons are taken into account. Recall that in SO(10) the 126 is needed to be able to fit the charged lepton masses. We saw that $\alpha\epsilon=1$ requires the physical t-quark mass to be less than about 45 GeV. If it turns out that m_t is greater than this value we can extend the allowed range of the model by relaxing the $\alpha\epsilon=1$ condition. This will allow a physical t-quark mass as large as about 95 GeV.[44] As discussed in section 3, the model is actually more predictive if $\alpha\epsilon\neq1$ in that δ is no longer a free parameter. However if m_t is less than 45 GeV we are likely to be close to $\alpha\epsilon=1$ in parameter space so, taking into account the experimental uncertainties in the various quantities and the sensitivity of the equations, the extra predictivity is somewhat illusory. As far as the characteristic predictions for neutrino masses and mixings are concerned there is not much difference between the $\alpha\epsilon=1$ and $\alpha\epsilon\neq1$ cases. Hence we shall concentrate the discussion in this

section mainly on finding the allowed ranges of the parameters for $\alpha\epsilon\neq1$.

We find the exact solutions of the set of equations (3.7) by numerical methods. We fix the quark masses as in section 5, allowing m_s and m_t to vary (holding $m_d/m_s=0.056$)[26]. It turns out that solutions exist only in certain regions which depend sensitively on the precise values (within the allowed experimental limits) of $|U_{12}|$ and $|U_{23}|$. The situation is summarized in Fig. 2 which shows the allowed regions for $|U_{12}|=.215$, .225, .235 and $|U_{23}|=.04$, .05, .06.; r is being taken as 3.00 but the regions are not sensitive to this choice. Let us consider holding $|U_{23}|$ fixed at its central value .05 and varying $|U_{12}|$. When $|U_{12}|$ takes on its central value of .225 we see that there are two nearly separate allowed regions. The curve corresponding to $\alpha\epsilon=1$ lies between them. At the point where the hour glass shaped disallowed region vanishes lies the $\alpha\epsilon=1$ type solution discussed in section 5. Notice that when $|U_{12}|=.215$ the upper allowed region shrinks while for $|U_{12}|=.235$, the lower allowed region shrinks. Fig. 2 illustrates that the effect of increasing $|U_{23}|$ is generally to allow a larger top quark mass.

As stated, the kind of neutrino physics which arises for $\alpha\epsilon\neq1$ is the same as that for the $\alpha\epsilon=1$ case discussed in section 5. The parameter δ which previously could be varied now turns out to be in the region of $\delta\approx0.5$ for most of the allowed areas in Fig. 2. Only very near $\alpha\epsilon=1$ can δ be close to zero. For a fixed value of m_s, increasing m_t causes $\alpha\epsilon$ to increase; $\alpha\epsilon$ is less than one in the lower allowed region and rises to a value between one and three in the upper allowed region. It is always reasonable to roughly approximate $\alpha\approx m_b/m_t$. We see from Fig. 2 that allowing $\alpha\epsilon\neq 1$ enables us to achieve a t-quark mass as high as about 165 GeV (at a scale of 1 GeV) which amounts to a physical value of about 95 GeV.

It may be worthwhile to present examples of the three neutrino scenarios previously discussed -- orthodox, magic canyon, and left-handed triplet dominance -- for a value of m_t larger than that allowed with $\alpha\epsilon=1$. First consider an orthodox scenario with the choices m_s=140 MeV and m_t (1 GeV) = 105 GeV. r is taken to be 3.00 to avoid the magic canyon. We may note that the ordinary K-M matrix[45] is here:

$$U = \begin{pmatrix} 0.97 & 0.225 & .0035@-102° \\ -0.22 & 0.97 & .050 \\ .012@-15° & .049@-179° & 1.00 \end{pmatrix}, \qquad (6.1)$$

which still gives a close to "maximal" invariant phase of 102°. The parameters of the model from fitting the quarks and charged leptons are α=.050, $\alpha\epsilon$=2.90, δ=.58, α_{12}=57°, α_{23}=3.9°. Choosing β=0, $\gamma=10^{12}$ for the orthodox see-saw case, we find the neutrino masses

$$m_1 = 4.72\times10^{-5}\ \text{eV}, \quad m_2 = 8.89\times10^{-3}\ \text{eV},$$

$$m_3 = 2.68\ \text{eV}\quad, \quad m_4 = 4.67\times10^{6}\ \text{GeV},$$

$$m_5 = 1.02\times10^{11}\ \text{GeV}, \quad m_6 = 1.35\times10^{11}\ \text{GeV}, \qquad (6.2)$$

and the mixing matrix,

$$K = \begin{pmatrix} 1.00 & .068@-148° & .014@-28° \\ .067@-30° & .98 & .19@-175° \\ .014@-98° & .19@-5° & .98 \end{pmatrix}. \qquad (6.3)$$

The pattern of (6.2) and (6.3) is similar to the one displayed by (5.7) and (5.8) in the $\alpha\epsilon$=1 case with a considerably smaller value of m_t. Notice that, even though δ is large here, the system is not close enough to the special values of δ for exotic behavior to take place.

Next consider a triplet dominance scenario with m_s=120 MeV, m_t (1 GeV) = 105 MeV and r=3.00. The parameters are α=.05, $\alpha\epsilon$=2.03, δ=0.65, α_{12}=53.7°, α_{23}=6.7°. The neutrino masses are given by

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = (10^9\beta) \begin{pmatrix} 7.5\times10^{-6} \\ 0.12 \\ 0.16 \end{pmatrix} \text{eV}, \qquad (6.5)$$

and they interact with charged leptons as dictated by the matrix

$$K = \begin{pmatrix} 1.00 & .053@-27° & .048@-140° \\ .070@-145° & .81 & .57@-93° \\ .014@4° & .57@-87° & .81 \end{pmatrix} . \qquad (6.6)$$

These are very similar to the corresponding equations (5.18) and (5.19) which hold for much smaller m_t.

Finally, consider a magic canyon scenario for a larger m_t. Take m_s=160 MeV, m_t (1 GeV)=85 GeV, r=3.065680 (slightly off the magic value) and $\gamma=10^{10}$. The parameters are α=.065, $\alpha\epsilon$=2.73, δ=0.51, α_{12}=55.7°, α_{23}=4.4°. There are two very light, two intermediate and two superheavy neutrinos:

$$m_1 = .0067 \text{ eV} , \ m_2 = .71 \text{ eV},$$

$$m_3 = 4.99 \text{ MeV} , \ m_4 = 15.8 \text{ GeV},$$

$$m_5 = m_6 = 1.11\times10^{9} \text{ GeV} . \qquad (6.7)$$

The first four neutrinos interact according to the matrix

$$K = \begin{pmatrix} 1.00 & .081@-134° & .014@-30° & .0003@-175° \\ .080@-44° & .98 & .19@-175° & .0034@-175° \\ .019@-97° & .19@-5° & .98 & .017@4° \end{pmatrix} . \qquad (6.8)$$

This parameter choice evidently corresponds to a physical situation similar to the one described by (5.13) and (5.14).

The present model, however, contains CP violating phases since $\delta\neq0$.

We have also investigated the consequences of using a value of r significantly different from 3. In particular for the choice r=4 we find that the allowed regions for the quark masses m_s and m_t in Fig. 2 hardly change at all. Furthermore the general structure of neutrino masses and mixings is also very similar to the examples given here for r=3. Because one is further away from magic r there will however be somewhat less enhancement (by about a factor 10) for the tau neutrino mass in the see-saw scenario.

7. Remarks on the solar neutrino problem

In the preceding we have studied the types of lepton mixing matrices and neutrino mass patterns which may arise in SO(10) with a plausible Ansatz for the mass matrices. The most spectacular possibility involves the "right-handed" neutrino mass matrix lying close to a singularity. The signature for this would be an abnormally heavy τ-neutrino; e.g. masses anywhere from the keV to MeV (rather than eV) range. From an a priori viewpoint this situation is not the most probable one since the "magic canyons" are extremely narrow. Nevertheless the possibility of such a scenario is just at the "cutting edge" of direct experimental test and is therefore quite interesting.

As stressed previously, the predictions for the neutrino mass ratios and the lepton mixing matrix in this model are actually very stable if one avoids the magic canyons. Since the lepton mixing matrix is almost uniquely predicted, assuming the conventional and plausible see-saw mechanism (i.e., B negligible) to hold, the results may be useful for the recent elegant explanations[30,31,46,47,48] of the "solar neutrino" problem. An important advantage of our model for the lepton mixing matrix is that it is computed in a framework which is consistent with the quark masses and mixing angles as well as the charged lepton masses. It is consistent with the recent bounds $|K_{12}|<0.2$ or $|K_{12}|>0.6$ found by LoSecco[49] who applied the idea of resonant matter oscillations to cosmic ray ν_μ's passing through the earth, with the assumption $m(\nu_\mu) \approx .01$ eV.

The problem[50] of too few relatively high energy electron neutrinos from the ^8B solar reaction reaching the ^{37}Cl detector on earth can be solved if the electron neutrino is resonantly transformed to another neutrino ν' with either of the following two relations[31,46] holding:

i) $m_{\nu'}^2 - m_e^2 \approx 10^{-4}\ (eV)^2$ and $|\theta| > .007$.

ii) $(m_{\nu'}^2 - m_e^2)\sin^2 2\theta \approx 3.2\times 10^{-8}\ (eV)^2$ and $(m_{\nu'}^2 - m_e^2) < 10^{-4}\ (eV)^2$.

The neutrino ν' can be either[47] a μ or a τ type neutrino. Depending on the mixing angle θ there may or may not be a

sizeable reduction in the number of neutrinos expected to be observed[46] in the proposed ^{71}Ga detector.

For our model, typical lepton mixing matrices are given in (5.8), (5.9) and (6.3). Notice that the forms of these matrices are essentially independent of the large scale parameter γ. Their differences are mostly due to different values of δ, which provides a measure of CP violation in the lepton system. Furthermore the neutrino masses scale as $1/\gamma$ so we have the rough expectations

$$m_1 \approx 2\text{-}5 \times (10^7/\gamma)\ \text{eV}$$

$$m_2 \approx 6\text{-}9 \times (10^9/\gamma)\ \text{eV}$$

$$m_3 \approx 3\text{-}8 \times (10^{12}/\gamma)\ \text{eV}\ .$$

Clearly m_1 is always negligible compared to the others.

First consider the case when the ν_e produced in the sun oscillates to the τ neutrino. We should identify θ with $|K_{13}|$ which is seen from (5.8), (5.9) or (6.3) to be around .010-.014. This is just large enough[46] for the resonant suppression mechanism in the ^{37}Cl experiments to take place. Actually both the possibilities i) and ii) above essentially coincide in this case. One needs $m_3 \approx .01$ eV, which corresponds to a scale parameter $\gamma \approx 10^{14}-10^{15}$. For this case the ^{71}Ga experiment is expected to show a negligible suppression according to the calculations of Rosen and Gelb.[46)]

Next consider the case when the ν_e produced in the sun oscillates to the μ neutrino. Here $|\theta|$ should be identified with $|K_{12}|$. We see that $|K_{12}|$ increases from about .01 for $\delta=0$ to .068 for $\delta=.58$. The examples (5.8) and (5.9) give a small $|K_{12}|$ for which the possibilities i) and ii) for suppression in the ^{37}Cl experiment essentially coincide and for which there would[46)] thus be negligible suppression in the ^{71}Ga experiment. The μ neutrino mass, m_2 would be about .01 eV corresponding to a scale $\gamma \approx 10^{12}$. Finally, for the example (6.3) where $|K_{12}|=.068$ the two possibilities i) and ii) give different physics. In i) we would again find $m_2 \approx .01$ eV ($\gamma=10^{12}$) and no suppression[46)] for the ^{71}Ga experiment while in ii) we would have a smaller μ neutrino mass $m_2 \approx 10^{-3}$ eV ($\gamma \approx 10^{13}$) together with a very sizeable suppression[46)] (by a factor of around five) in the ^{71}Ga experiment.

Remember that the parameter δ which distinguishes the various cases above is a free parameter only for the $\alpha\epsilon=1$ models. This corresponds to a physical top quark mass less than about 45 GeV. If the top quark mass is somewhat larger than this (as would be required[1,2)] for example to fit the CP violation parameter ϵ in the K_L-K_S system with a "bag constant" B of the order of one half rather than of the order unity) we would find δ automatically by specifying m_t. It would, as we

have discussed, tend to be similar to its value in the example (6.3). If one assumes this situation to be the likely one then the amount of suppression in the ^{71}Ga experiment would depend on the scale γ. If $\gamma \approx 10^{14}$-10^{15} or if $\gamma \approx 10^{12}$ one would not expect any suppression in this model. On the other hand $\gamma \approx 10^{13}$ would lead to a marked suppression. All these values of γ are quite reasonable as scales associated with grand unification. It is clear that the experimental study of neutrino physics may be a valuable tool in understanding the structure of particle theories at the GUT scale.

A preliminary version of this work has been submitted elsewhere.[51)]

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Figure Captions

Fig. 1 Allowed (unshaded) region in the m_s-m_t plane for the $\alpha\epsilon=1$ (exact Stech-Fritzsch model) case. The axes are labelled by the running masses at 1 GeV; physical masses are shown in parentheses. We have fixed $m_s/m_d=18$, $m_u/m_d=0.57$ and required $|U_{us}|$ to be in the range 0.225±.01 and $|U_{cb}|$ to be in the range 0.05±.01.

Fig. 2 Allowed (unshaded) regions in the m_s-m_t plane in the general $\alpha\epsilon\neq1$ case for various fixed values of $|U_{us}|$ in the range 0.225±.01 and $|U_{cb}|$ in the range 0.05±.01. The axes are labelled as in the central plot by the running masses at 1 GeV.

Fig. 1

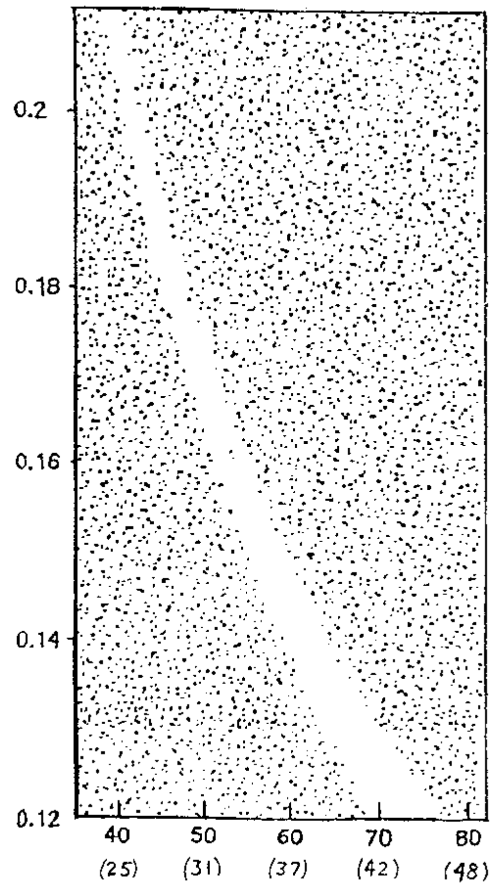
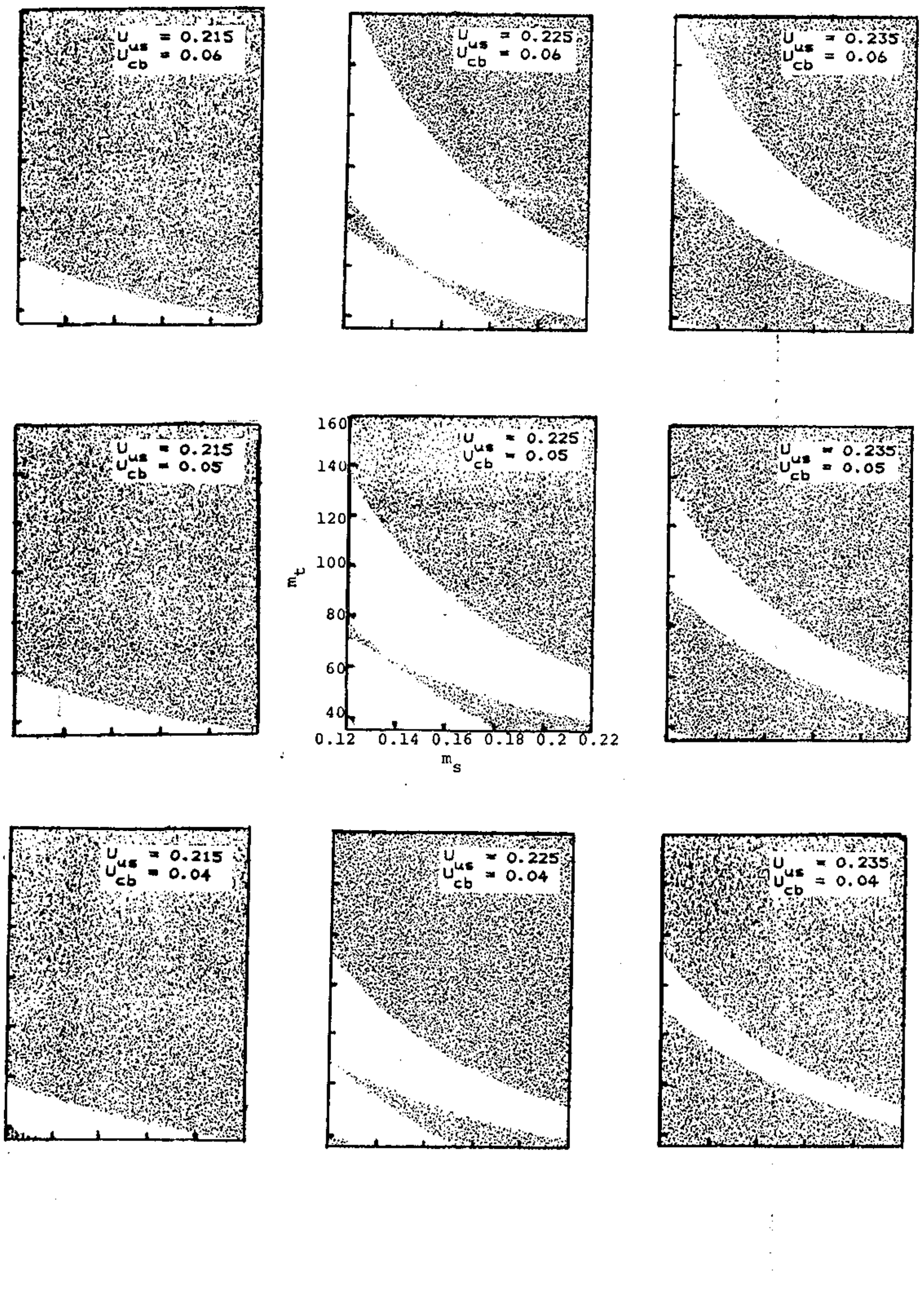


Fig. 2



U_us = 0.215
U_cb = 0.06
U_us = 0.225
U_cb = 0.06
U_us = 0.235
U_cb = 0.06
U_us = 0.215
U_cb = 0.05
U_us = 0.225
U_cb = 0.05
U_us = 0.235
U_cb = 0.05
160
140
120
100
80
60
40
m_t
0.12
0.14
0.16
0.18
0.2
0.22
m_s
U_us = 0.215
U_cb = 0.04
U_us = 0.225
U_cb = 0.04
U_us = 0.235
U_cb = 0.04